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On the $\mu x^2 + \lambda x^4$ interaction

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Abstract. The even and odd parity eigenvalues for the bounded potential $\mu x^2 + \lambda x^4$, with both positive and negative values of μ and $\lambda > 0$, are obtained by the method of series solution.

The energy eigenvalues of the anharmonic oscillator with anharmonicity λx^4 have been studied extensively by a number of authors (Banerjee *et al* 1978, Bender and Wu 1969, Loeffel *et al* 1970, Fung *et al* 1978, Drummond 1981, Biswas *et al* 1971, 1973, Halpern 1973, Singh *et al* 1978, Bozzolo *et al* 1982, Killingbeck 1978, Austin and Killingbeck 1982) using different perturbative and non-perturbative techniques. Bender and Wu (1969) have shown that the perturbation series in terms of the parameter λ is divergent for all λ though each term of the series is finite. The Padé approximation method and the Borel summation technique have been used to recover finite results for the energy correction (Simon 1970, Graffi *et al* 1970, Graffi and Grecchi 1978, Loeffel *et al* 1970). The logarithmic perturbation expansion (Dolgov *et al* 1980, Aharonov and Au 1979) has also been used to study the λx^4 anharmonic oscillator. The first-order logarithmic perturbation iteration method (Au *et al* 1983, Au 1980) which introduces cut-offs in the numerical integrations is, however, not very accurate in evaluating the excited state energy eigenvalues. The eigenvalues of the anharmonic oscillator of type λx^{2m} have been calculated by Biswas *et al* (1971, 1973) using the infinite Hill determinant method which produces the eigenvalues to a high degree of accuracy for any arbitrary value of the coupling constant λ. The energy eigenvalues of the anharmonic oscillators are also obtained in a semi-empirical manner (Hioe and Montroll 1975, Hioe *et al* 1978, Mathews *et al* 1981) using the extended WKB formula.

Recently the pure λx^{2m} oscillators bounded by infinite potentials at $x = \pm L$ have been studied (Chaudhuri and Mukherjee 1983, Barakat and Rosner 1981) by the method of series solution, and it has been shown that the lower-order eigenvalues tend rapidly to the values of the unbounded oscillator as L is made larger. This finite box method is applied here to the case of anharmonic oscillator with anharmonicity λx^4 bounded by infinitely high potentials at $x = \pm L$. We have to solve the eigenvalue equation

$$(-\mathrm{d}^2/\mathrm{d}x^2 + \mu x^2 + \lambda x^4)\psi(x) = E\psi(x) \tag{1}$$

subject to the boundary conditions $\psi(\pm L) = 0$. We make the change of variable $y = x/L$

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so that the boundary conditions become $\psi(y=\pm 1)=0$ and equation (1) is transformed to

$$(\mathrm{d}^2/\mathrm{d}y^2+\varepsilon-ay^2-by^4)\psi(y)=0 \tag{2}$$

where $\varepsilon=EL^2$, $a=\mu L^4$ and $b=\lambda L^6$.

It is clear from (2) that $y=0$ is an ordinary point and $y=\infty$ is an irregular singular point of the differential equation. The symmetry of equation (2) implies that $\psi(y)$ admits a convergent even and odd power series solution valid in the box $|x|\leqslant L$:

$$\psi^{\mathrm{e}}(y)=\sum_{n=0}^{\infty} A_{2n}y^{2n} \tag{3}$$

$$\psi^{\mathrm{o}}(y)=\sum_{n=0}^{\infty} A_{2n+1}y^{2n+1} \tag{4}$$

where the superscripts 'e' and 'o' refer to the even and odd parity solutions. It should be noted that the replacement of n by $n+\frac{1}{2}$ in the even series solution reproduces the odd series solution. The coefficients A_{2n} and A_{2n+1} satisfy a set of recursion relations. The recursion relations satisfied by the even coefficients are given below:

$$2A_2+\varepsilon A_0=0 \tag{5a}$$

$$12A_4+\varepsilon A_2-aA_0=0 \tag{5b}$$

$$2n(2n-1)A_{2n}+\varepsilon A_{2n-2}-aA_{2n-4}-bA_{2n-6}=0 \qquad n\geqslant 3. \tag{5c}$$

Similar relations exist for the odd coefficients.

We put $A_0=A_1=1$ and apply the recursion relations to evaluate

$$f^{\mathrm{e}}(\varepsilon)=\sum_{n=0}^{\infty} A_{2n} \qquad f^{\mathrm{o}}(\varepsilon)=\sum_{n=0}^{\infty} A_{2n+1}. \tag{6}$$

The zeros of the functions $f^{\mathrm{e,o}}(\varepsilon)$ give us the eigenvalues of the even and odd parity solutions. In table 1 we tabulate the first four eigenvalues of the confined $\mu x^2+\lambda x^4$ anharmonic oscillator with $\mu=1$, $\lambda=0.1$, 1.0, 10.0, 100.0 and $L=1$, 2, 3. Our values for $\mu=1$ with $L=3$ are also compared in table 1 with the exact eigenvalues for the unbounded oscillator (Biswas *et al* 1973) and it is found that the agreement is excellent. The results may be improved further by increasing L and by including a large number of terms of equations (6). It is clear from the table that when λ is large the eigenvalues remain almost unchanged for $L=2$ and 3 where the potentials are made infinitely high. This is because for large λ the potential function $\mu x^2+\lambda x^4$ becomes effectively infinite at $x=2$ in comparison with its values around $x=0$. This method of finite box approximation is suitable for finding the eigenvalues of the unbounded oscillator when λ is large.

The variation of the eigenvalues for the potential $\mu x^2+\lambda x^4$ confined in a box for different positive and negative values of μ and fixed λ is shown in table 2. It should be mentioned here that for negative μ the term λx^4 cannot be treated as a distortion over the harmonic oscillator. For the harmonic oscillator problem we have the convergence factor $\exp(-\frac{1}{2}\sqrt{\mu}x^2)$ which demands that μ should be positive for the normalisation of the wavefunction. The Hill determinant method of Biswas *et al* (1971) also fails for negative μ. For negative μ one has to find a completely new wavefunction which is square integrable, otherwise one may get the wrong answer (Saxena and Varma 1982). In our method the advantage is that the eigenvalues are obtained from

Table 1. The first four eigenvalues of the confined $\mu x^2 + \lambda x^4$ oscillator with $\mu = 1$, $\lambda = 0.1$, 1.0, 10.0, 100.0 and $L = 1$, 2, 3 and those of the unbounded ($L \to \infty$) oscillator (Biswas *et al* 1973).

λ	$L=1$	$L=2$	$L=3$	Unbounded oscillator ($L \to \infty$)
0.1	2.601	1.116	1.065	1.0653
	10.162	3.673	3.309	3.3069
	22.534	7.036	5.761	5.7480
	39.817	11.452	8.417	8.3527
1.0	2.637	1.398	1.392	1.3924
	10.263	4.695	4.649	4.6488
	22.675	8.868	8.655	8.6550
	39.975	13.836	13.157	13.1568
10.0	2.969	2.449	2.449	2.4492
	11.228	8.599	8.599	8.5990
	24.068	16.636	16.636	16.6359
	41.555	25.806	25.806	25.8063
100.0	4.989	4.999	4.999	4.9994
	17.946	17.830	17.830	17.8302
	35.399	34.874	34.874	34.8740
	56.116	54.385	54.385	54.3853

Table 2. The first four eigenvalues for the bounded potential $\mu x^2 + \lambda x^4$ with $\lambda = 1$, $\mu = -1.0$, 0, 1.0, 4.0 and $L = 1$, 2, 3 and those of the unbounded ($L \to \infty$) oscillator (Chan and Stellman (1963) for $\mu = 0$ and Biswas *et al* (1973) for $\mu = 1.0$).

μ	$L=1$	$L=2$	$L=3$	Unbounded oscillator ($L \to \infty$)
−1.0	2.377	0.687	0.658	
	9.701	2.985	2.835	
	22.054	6.665	6.164	
	39.333	11.323	10.038	
0	2.508	1.073	1.060	1.0604
	9.983	3.882	3.800	3.7997
	22.364	7.784	7.456	7.4557
	39.654	12.584	11.645	11.6448
1.0	2.637	1.398	1.392	1.3924
	10.263	4.695	4.649	4.6488
	22.675	8.868	8.655	8.6550
	39.975	13.836	13.157	13.1568
4.0	3.008	2.160	2.159	
	11.090	6.748	6.740	
	23.609	11.845	11.787	
	40.942	17.457	17.209	

a single equation for both positive as well as negative values of μ so long as λ is positive. When λ is positive the potential function $V(x)=\mu x^2+\lambda x^4$ remains confining for both positive and negative μ. Our method of finite box approximation is not, however, applicable for negative values of λ. This is because we are making the potentials $+\infty$ at $x=\pm L$ while the potential function $\mu x^2+\lambda x^4$ tends to $-\infty$ with increasing x for negative λ. The problem of stabilisation (Hazi and Taylor 1970) may set in for negative λ as is in the renormalisation series approach (Killingbeck 1981, Austin and Killingbeck 1982). Recently Flessas *et al* (1983) have obtained a class of exact solutions for the Schrödinger equation with the potential $V(x)=\mu x^2+\lambda x^4$, $x>0$, $\mu \geqslant 0$, $\lambda<0$, and have shown that the corresponding energy spectrum is continuous. They have, however, used a cut-off in the potential function for large x.

The problem of applying the series method for the unbounded oscillator has been discussed by us in a recent paper (Chaudhuri and Mukherjee 1983). For the unbounded oscillator the boundary conditions $\psi(\pm\infty)=0$ cannot be imposed on the infinite series since the point at infinity is an irregular singular point of the differential equation (1) and the series may not be convergent at that point. For the bounded oscillator the boundary conditions $\psi(\pm L)=0$ do not pose any problem. For the unbounded eigenvalue problem one has to find a proper convergence x-factor for the wavefunction (Killingbeck 1981, Ginsburg 1982). The convergence factor is not required for the evaluation of the energy eigenvalues of the bounded potential problem by the infinite series method. It can be shown easily that the Hill determinant diverges if the proper convergence x-factor is not factored out from the wavefunction. Moreover, the Hill determinant method or the equivalent infinite continued fraction method may give rise to spurious solutions (Chaudhuri 1983, Fleassas 1982) since the boundary conditions $\psi(\pm\infty)=0$ are not incorporated into the method.

Our method of finite box approximation is applicable to any form of potentials having no singularity for finite values of x. An oscillator with λx^{2n} distortion will be discussed in the future. The finite box method is also applicable for the potential $V(x)=\mu x^2+\lambda x^4+\eta x^6$ with positive η and positive or negative λ and μ. This potential has been discussed by a number of authors (Khare 1981, Flessas and Das 1980) in the context of negative λ. A particular feature of the eigenvalues of this potential is that Rayleigh–Schrödinger perturbation theory may not be applicable for negative λ. The potential with positive and negative λ will be studied in the future by the method of finite box approximation.

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